Normal and Bivariate Normal Distributions

and Moment-Generating Functions

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Given a random variable X with probability density function $f_X(x)$ the moment-generating function of X is given by [1]

$$M(t) = E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx, \tag{1}$$

for values of t such that the integral converges. It has the property that

$$M(t=0) = 1, \qquad \left.\frac{dM}{dt}\right|_{t=0} = \mu_x, \qquad \left.\frac{d^2M}{dt^2}\right|_{t=0} = E\left(X^2\right), \tag{2}$$

where μ_x is the mean of X. In general the m^{th} derivative of $M(t)$ evaluated at $t = 0$ gives the expectation of X^m.

For the normal distribution

$$f_X(x) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma_x^2}(x-\mu_x)^2\right], \tag{3}$$

the moment-generating function satisfies

$$M(t) = \frac{1}{\sigma_x\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[tx - \frac{(x-\mu_x)^2}{2\sigma_x^2}\right] dx. \tag{4}$$

To evaluate the integral in Eq. (4) note that

$$tx - \frac{(x-\mu_x)^2}{2\sigma_x^2} = -\frac{1}{2\sigma_x^2}\left\{\left[x - \left(\mu_x + \sigma_x^2 t\right)\right]^2 - \left(\mu_x + \sigma_x^2 t\right)^2 + \mu_x^2\right\}, \tag{5}$$

so

$$M(t) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp\left[-\frac{\mu_x^2 - (\mu_x + \sigma_x^2 t)^2}{2\sigma_x^2}\right] \int_{-\infty}^{\infty} \exp\left\{-\frac{[x - (\mu_x + \sigma_x^2 t)]^2}{2\sigma_x^2}\right\} dx. \tag{6}$$

The integral in Eq. (6) evaluates to $\sigma_x\sqrt{2\pi}$ so

$$M(t) = \exp\left[-\frac{\mu_x^2 - (\mu_x + \sigma_x^2 t)^2}{2\sigma_x^2}\right] = \exp\left(\mu_x t + \frac{\sigma_x^2 t^2}{2}\right). \tag{7}$$

From Eq. (7) we find

$$M\left(t=0\right)=1,\qquad \left.\frac{dM}{dt}\right|_{t=0}=\mu_x,\qquad \left.\frac{d^2M}{dt^2}\right|_{t=0}=\sigma_x^2+\mu_x^2, \tag{8}$$

as expected.

Bivariate Normal Density

If two random variables X and Y are jointly normal their density function satisfies [2]

$$f_{XY}\left(x,y\right)=B\exp\left\{-\frac{1}{2\left(1-r^2\right)}\left[\frac{\left(x-\mu_1\right)^2}{\sigma_1^2}-\frac{2r\left(x-\mu_1\right)\left(y-\mu_2\right)}{\sigma_1\sigma_2}+\frac{\left(y-\mu_2\right)^2}{\sigma_2^2}\right]\right\}, \tag{9}$$

where B is to be determined from

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{XY}\left(x,y\right)\,dx\,dy=1. \tag{10}$$

Also, $|r|<1$ and the significance of r, μ_1, μ_2, σ_1, σ_2 are to be determined.

To evaluate $f_Y\left(y\right)$ use

$$\frac{\left(x-\mu_1\right)^2}{\sigma_1^2}-\frac{2r\left(x-\mu_1\right)\left(y-\mu_2\right)}{\sigma_1\sigma_2}+\frac{\left(y-\mu_2\right)^2}{\sigma_2^2}=\left[\frac{x-\mu_1}{\sigma_1}-\frac{r\left(y-\mu_2\right)}{\sigma_2}\right]^2+\frac{\left(1-r^2\right)\left(y-\mu_2\right)^2}{\sigma_2^2}, \tag{11}$$

$$=\frac{1}{\sigma_1^2}\left[x-\mu_1-\frac{r\sigma_1\left(y-\mu_2\right)}{\sigma_2}\right]^2+\frac{\left(1-r^2\right)\left(y-\mu_2\right)^2}{\sigma_2^2}, \tag{12}$$

so that

$$f_Y\left(y\right)=B\exp\left[-\frac{\left(y-\mu_2\right)^2}{2\sigma_2^2}\right]\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2\sigma_1^2\left(1-r^2\right)}\left[x-\mu_1-\frac{r\sigma_1\left(y-\mu_2\right)}{\sigma_2}\right]^2\right\}dx. \tag{13}$$

The integral in Eq. (13) evaluates to $\sqrt{2\pi\sigma_1^2\left(1-r^2\right)}$ so

$$f_Y\left(y\right)=B\sigma_1\sqrt{2\pi\left(1-r^2\right)}\exp\left[-\frac{\left(y-\mu_2\right)^2}{2\sigma_2^2}\right], \tag{14}$$

showing that $f_Y\left(y\right)$ is a normal distribution with mean μ_2 and variance σ_2^2. This means that

$$B\sigma_1\sqrt{2\pi\left(1-r^2\right)}=\frac{1}{\sigma_2\sqrt{2\pi}}, \tag{15}$$

or

$$B = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}}. \tag{16}$$

Given the symmetry of the variables in Eq. (9) we now have

$$f_Y(y) = \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left[-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right], \tag{17}$$

$$f_X(x) = \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left[-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right], \tag{18}$$

and so may conclude that μ_1, μ_2, σ_1^2, σ_2^2 are respectively the x and y means and the x and y variances of the distribution of Eq. (9). Accordingly,

$$\mu_1 = \mu_x, \qquad \mu_2 = \mu_y, \qquad \sigma_1^2 = \sigma_x^2, \qquad \sigma_2^2 = \sigma_y^2, \tag{19}$$

are now used in the p.d.f. in Eq. (9) yielding

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-r^2}}$$

$$\times \exp\left\{-\frac{1}{2(1-r^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} - \frac{2r(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right\}. \tag{20}$$

It remains to demonstrate the significance of r in Eq. (20). To accomplish this the covariance of X and Y defined [3]

$$E[(X-\mu_x)(Y-\mu_y)] = E(XY) - \mu_x\mu_y, \tag{21}$$

and the correlation coefficient of X and Y

$$\rho_{xy} = \frac{E(XY) - \mu_x\mu_y}{\sigma_x\sigma_y}, \tag{22}$$

are introduced. Rather than calculating $E(XY)$ directly this quantity is computed in a series of steps that will assist in calculating the moment-generating function for the bivariate p.d.f. later. The first step is to evaluate

$$f_Y(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}. \tag{23}$$

Substituting Eqs. (18) and (20) into Eq. (23) yields

$$f_Y(y|x) = \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-r^2}}$$

$$\times \exp\left\{-\frac{1}{2\left(1-r^2\right)}\left[\frac{\left(x-\mu_x\right)^2}{\sigma_x^2}-\frac{2r\left(x-\mu_x\right)\left(y-\mu_y\right)}{\sigma_x\sigma_y}+\frac{\left(y-\mu_y\right)^2}{\sigma_y^2}\right]+\frac{\left(x-\mu_x\right)^2}{2\sigma_x^2}\right\}, \tag{24}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-r^2}}\exp\left\{-\frac{1}{2\left(1-r^2\right)}\left[\frac{\left(x-\mu_x\right)^2 r^2}{\sigma_x^2}-\frac{2r\left(x-\mu_x\right)\left(y-\mu_y\right)}{\sigma_x\sigma_y}-\frac{\left(y-\mu_y\right)^2}{\sigma_y^2}\right]\right\}, \tag{25}$$

or

$$f_Y\left(y|x\right) = \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-r^2}}\exp\left\{-\frac{1}{2\left(1-r^2\right)}\left[\frac{\left(y-\mu_y\right)}{\sigma_y}-\frac{r\left(x-\mu_x\right)}{\sigma_x}\right]^2\right\}. \tag{26}$$

Equation (26) may be written

$$f_Y\left(y|x\right) = \frac{1}{\sqrt{2\pi}\sigma_y\sqrt{1-r^2}}\exp\left\{-\frac{1}{2\sigma_y^2\left(1-r^2\right)}\left[y-\mu_y-\frac{r\sigma_y\left(x-\mu_x\right)}{\sigma_x}\right]^2\right\}, \tag{27}$$

so that $f_Y\left(y|x\right)$ is also a normal density in y for fixed x with mean

$$\mu_{y|x} = \mu_y + \frac{r\sigma_y}{\sigma_x}\left(x-\mu_x\right), \tag{28}$$

and variance

$$\sigma_{y|x}^2 = \sigma_y^2\left(1-r^2\right). \tag{29}$$

Now $E\left(XY\right)$ is determined as follows [4]. Equation (28) shows that

$$\int_{-\infty}^{\infty} y\,\frac{f_{XY}\left(x,y\right)}{f_X\left(x\right)}\,dy = \mu_y + \frac{r\sigma_y}{\sigma_x}\left(x-\mu_x\right), \tag{30}$$

or

$$\int_{-\infty}^{\infty} y f_{XY}\left(x,y\right)\,dy = \left[\mu_y + \frac{r\sigma_y}{\sigma_x}\left(x-\mu_x\right)\right] f_X\left(x\right). \tag{31}$$

However,

$$E\left(XY\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}\left(x,y\right)\,dx\,dy, \tag{32}$$

$$= \int_{-\infty}^{\infty}\left[\mu_y + \frac{r\sigma_y}{\sigma_x}\left(x-\mu_x\right)\right] x f_X\left(x\right)\,dx. \tag{33}$$

So

$$E\left(XY\right) = \mu_x\mu_y - \frac{r\sigma_y}{\sigma_x}\mu_x^2 + \frac{r\sigma_y}{\sigma_x}E\left(X^2\right), \tag{34}$$

$$= \mu_x \mu_y - \frac{r\sigma_y}{\sigma_x}\mu_x^2 + \frac{r\sigma_y}{\sigma_x}\left(\sigma_x^2 + \mu_x^2\right), \tag{35}$$

or

$$E\left(XY\right) = \mu_x \mu_y + r\sigma_x\sigma_y. \tag{36}$$

Now it is seen from Eq. (22)

$$\rho_{xy} = r, \tag{37}$$

displaying that the factor r appearing in Eq. (20) is the correlation coefficient of the random variables X and Y.

Moment-Generating Function for Bivariate Normal Distribution

The moment-generating function for two random variables X and Y is defined [5]

$$M\left(t_x, t_y\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(t_x x + t_y y\right) f_{XY}\left(x, y\right)\, dx\, dy. \tag{38}$$

Using Eq. (23) this becomes

$$M\left(t_x, t_y\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left(t_x x + t_y y\right) f_X\left(x\right) f_Y\left(y|x\right)\, dx\, dy, \tag{39}$$

or

$$M\left(t_x, t_y\right) = \int_{-\infty}^{\infty} \exp\left(t_x x\right) f_X\left(x\right) \left[\int_{-\infty}^{\infty} \exp\left(t_y y\right) f_Y\left(y|x\right)\, dy\right] dx. \tag{40}$$

Now the inner integral within the bracket of Eq. (40) is immediately recognized as the moment-generating function of the normal density function given in Eq. (27). The result for the moment-generating function given in Eq. (7) is repeated here for convenience

$$M_{\text{normal}}\left(t; \mu, \sigma\right) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right). \tag{41}$$

The inner integral within the bracket of Eq. (40) is now immediately evaluated as

$$\int_{-\infty}^{\infty} \exp\left(t_y y\right) f_Y\left(y|x\right)\, dy$$

$$= M_{\text{normal}}\left(t_y; \mu_{y|x}, \sigma_{y|x}\right) = \exp\left[\mu_y t_y + \frac{r\sigma_y}{\sigma_x}\left(x - \mu_x\right) t_y + \frac{\sigma_y^2 t_y^2}{2}\left(1 - r^2\right)\right]. \tag{42}$$

when the results of Eqs. (28) and (29) are used.

Thus,

$$M(t_x, t_y) = \exp\left[\mu_y t_y - \frac{r\sigma_y}{\sigma_x}\mu_x t_y + \frac{\sigma_y^2 t_y^2}{2}\left(1 - r^2\right)\right] \int_{-\infty}^{\infty} \exp\left[\left(t_x + \frac{r\sigma_y}{\sigma_x}t_y\right)x\right] f_X(x)\, dx. \tag{43}$$

The integral in Eq. (43) is now recognized as

$$N_{\text{normal}}\left(t_x + \frac{r\sigma_y}{\sigma_x}t_y;\, \mu_x, \sigma_x\right) = \exp\left[\mu_x\left(t_x + \frac{r\sigma_y}{\sigma_x}t_y\right) + \frac{\sigma_x^2}{2}\left(t_x + \frac{r\sigma_y}{\sigma_x}t_y\right)^2\right], \tag{44}$$

yielding

$$M(t_x, t_y) = \exp\left[\mu_y t_y + \frac{\sigma_y^2 t_y^2}{2}\left(1 - r^2\right) + \mu_x t_x + \frac{\sigma_x^2}{2}\left(t_x + \frac{r\sigma_y}{\sigma_x}t_y\right)^2\right], \tag{45}$$

or

$$M(t_x, t_y) = \exp\left[\mu_x t_x + \mu_y t_y + \frac{1}{2}\left(\sigma_x^2 t_x^2 + 2r\sigma_x\sigma_y t_x t_y + \sigma_y^2 t_y^2\right)\right]. \tag{46}$$

This result will be important in the discussion of multivariate distributions.

Matrix Form of Bivariate Normal Density Function

Hogg and Craig [6] point out that the bivariate normal p.d.f. of Eq. (20) may be written in a compact matrix form. We shall display the form they propose here and compare it with the form found in Eq. (20). In compact form

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{|\mathbf{V}|}} \exp\left[-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \mathbf{V}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right], \tag{47}$$

where

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \boldsymbol{\mu} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \qquad \mathbf{V} = \begin{bmatrix} \sigma_x^2 & r\sigma_x\sigma_y \\ r\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}, \tag{48}$$

and $|\mathbf{V}|$ is the determinant of $\mathbf{V}$. Since the determinant of $\mathbf{V}$ satisfies $|\mathbf{V}| = \sigma_x^2\sigma_y^2(1 - r^2)$ the coefficient of the exponential in Eq. (47) agrees with the coefficient given in Eq. (20). Since

$$\mathbf{V}^{-1} = \frac{1}{\sigma_x^2\sigma_y^2(1 - r^2)}\begin{bmatrix} \sigma_y^2 & -r\sigma_x\sigma_y \\ -r\sigma_x\sigma_y & \sigma_x^2 \end{bmatrix}, \tag{49}$$

the argument of the exponential in Eq. (47) satisfies

$$(\mathbf{x}-\boldsymbol{\mu})^T \mathbf{V}^{-1} (\mathbf{x}-\boldsymbol{\mu})$$

$$= \frac{1}{\sigma_x^2\sigma_y^2\left(1-r^2\right)} \begin{bmatrix} x-\mu_x & y-\mu_y \end{bmatrix} \begin{bmatrix} \sigma_y^2 & -r\sigma_x\sigma_y \\ -r\sigma_x\sigma_y & \sigma_x^2 \end{bmatrix} \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}, \tag{50}$$

$$= \frac{1}{1-r^2}\left[\frac{(x-\mu_x)^2}{\sigma_x^2} - \frac{2r\,(x-\mu_x)\,(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right], \tag{51}$$

in agreement with Eq. (20) and demonstrating that Eqs. (20) and (47) are the same.

Finally, given the definition of Eq. (22) and the result $\rho_{xy} = r$

$$r\sigma_x\sigma_y = E\left[(x-\mu_x)\,(y-\mu_y)\right], \tag{52}$$

and since

$$\sigma_x^2 = E\left[(x-\mu_x)^2\right], \qquad \sigma_y^2 = E\left[(y-\mu_y)^2\right], \tag{53}$$

we have

$$\mathbf{V} = E\left[(\mathbf{x}-\boldsymbol{\mu})\,(\mathbf{x}-\boldsymbol{\mu})^T\right] = E\left(\mathbf{x}\mathbf{x}^T\right) - \boldsymbol{\mu}\boldsymbol{\mu}^T, \tag{54}$$

a compact result.

References

[1] R. V. Hogg and A.T. Craig, *Introduction to Mathematical Statistics*, Macmillan, N.Y. (1978), pp. 50 - 52.

[2] A. Papoulis, *Probability, Random Variables and Stochastic Processes*, McGraw-Hill, N.Y. (1965), pp. 182 - 184.

[3] Ibid., pp. 209 - 210.

[4] Ibid., Hogg and Craig, pp. 74 - 75.

[5] Ibid., pp. 119 - 120.

[6] Ibid., p. 409.